Pro domo sua

Grzegorz Łysik

Jan Kochanowski University, Kielce, Poland

e-mail: glysik@ujk.edu.pl

Lecture on the seminar
of the Institute of Physics UJK

Kielce, June 21, 2023

1. Introduction

One of the main problems arising in the analytic theory of partial differential equations is a characterization of data given on a manifold S for which a solution of a boundary value problem is an analytic function in a variable normal to S. In general, one can easily obtain formal power series solutions in a variable normal to S, and by the Cauchy-Kowalevski theorem it is convergent if S is not the characteristic variety of the equation.

In other cases formal solutions need not to be convergent. At this point there arise natural questions:

- under which conditions on the data the formal solution is convergent;
- what is the meaning of a formal solution;
- is it an asymptotic expansion of an actual solution;
- can and how the actual solution be constructed from the formal one.

In case of ordinary differential equations the answers to those questions were given in 80-ties and 90-ties of the XX century by the (multi)summability theory. On the other hand in the case of partial differential equations the study of those problems started at the end of the XX century.

In the lecture we shall give a survey of solutions to those problems for some classes of partial differential equations. A special attention is put on results obtained by ourselves concerning convergence and Borel summability of formal solutions of the heat equation and its generalizations.
Finally we shall introduce a definition of analytic functions on metric measure spaces and state that they satisfy some uniqueness property.

2. One dimensional case.

The starting point in the study of summability of formal solutions to PDE's is the paper by Lutz, Miyake and Schäfke [LMS-99]. They studied initial value problem to the one dimensional heat equation

$$(1)\begin{cases} \partial_t u - \partial_z^2 u &= 0, \\ u_{|t=0} &= u_0 \in \mathcal{O}(B), \quad B \text{ a ball in } \mathbb{C}. \end{cases}$$

Its formal power series solution $\widehat{u}$ is given by

$$(2) \qquad \widehat{u}(t,z) = \sum\nolimits_{j=0}^{\infty} \frac{\partial^{2j} u_0(z)}{j!} t^j .$$

In general the series $\widehat{u}$ is divergent, but Gevrey of order $k = 1$, i.e.,

$$\left|\frac{\partial^{2j} u_0(z)}{j!}\right| \leq C^{j+1}(j!)^k \quad \text{with} \quad k = 1,$$

loc. uni. in B.

The problem of a characterization of initial data ensuring convergence was solved already by Kowalevska in [Kow 1875]. She proved that the solution $\widehat{u}$ is convergent if and only if the initial data u_0 can be analytically extended to an entire function of exponential order 2, i.e. $|u_0(z)| \leq C \exp\{c|z|^2\}$.

To state the main result of [LMS-99] recall

Definition. Let $d \in \mathbb{R}$, $U \subset \mathbb{C}^n$ and $\widehat{\varphi}_j \in \mathcal{O}(U)$. A formal power series

$$\widehat{\varphi}(t,z) = \sum\nolimits_{j=0}^{\infty} \frac{\varphi_j(z)}{j!} t^j$$

is 1-*summable* (or *Borel summable*) with respect to t in the direction d if its Borel transform defined on $B_\varepsilon \times U$ by

$$\left(\widehat{\mathcal{B}}\widehat{\varphi}\right)(s,z) = \sum\nolimits_{j=0}^{\infty} \frac{\varphi_j(z)}{j! \cdot \Gamma(1+j)} s^j$$

extends holomorphically to a domain

$\big(B_\epsilon \cup S(d,\epsilon)\big) \times U$ with some $0 < \epsilon$
and the extension satisfies

$$\sup_{z\in U_1} |\big(\widehat{\mathcal{B}}\widehat{\varphi}\big)(s,z)| \leq Ae^{B|s|} \quad \text{for} \quad s \in S(d,\epsilon_1)$$

with some $A, B < \infty$, $\quad (U_1 \Subset U$ and $0 < \epsilon_1 < \epsilon)$.

If so, then the function

$$\varphi^\theta(t,z) = \frac{1}{t}\int_0^{\infty(\theta)} \widehat{\mathcal{B}}\widehat{\varphi}(s,z)e^{-(s/t)}\,ds$$

is called the 1-Borel sum of $\widehat{\varphi}$.

Theorem LMS (Lutz, Miyake, Schäfke, 1999). *Let* $u_0 \in \mathcal{A}(B)$, B *a ball at* $\{0\}$.
The formal power series solution (2) *of* (1) *is Borel summable in a direction d loc. uni. in B iff* u_0 *extends analytically to a function holomorphic on a domain*

$$D(d, \varepsilon) \supset S(d/2, \varepsilon) \cup S(d/2 + \pi, \varepsilon)$$

with some $\varepsilon > 0$ *which has in* $D(d, \varepsilon)$ *at most exponential growth of order at most* 2 *loc. uni. in* B.

The result was extended to the case of multi-summable solutions of (1) by Balser [B-99];
to formal power series satisfying certain differential recursion formulas by Balser, Miyake [BM-99];
to the equation $\partial_t^p u = \partial_z^q u$, $p < q$, by Miyake [Miy-99]
and by Ichinobe [I-01], who also gave explicite integral representations of the Borel sums of solutions in terms of the Barnes hypergeometric series ${}_qF_{p-1}$.

General linear partial differential equations with constant coefficients in one space variable

$$\big(\partial_t^m p(\partial_z) - \sum\nolimits_{i=1}^{m} \partial_t^{m-i} p_i(\partial_z)\big)u = 0,$$

where p and p_i are polynomials, were investigated by Balser. In [B-02] he studied the case when the Newton polygon of the equation has only one slope and proved k-summability of a (unique) normalized solution. While in [B-04] he proved multisummability of normalized solutions to equations with Newton polygon having several slopes.

The results were further extended in [B-05] to solutions of some integral-differential equations in two variables. Another proof of Balser's results in a more general framework of fractional equations was given by Michalik [M-10].

In [I-03] Ichinobe studied the following problem

$$\begin{cases} \partial_t^{p\nu} u & = \sum_{j=1}^{\nu} a_j \partial_t^{p(\nu-j)} \partial_z^{jq} u, \quad q > p \geq 1, \\ \partial_t^k u_{|t=0} & = 0 \ \text{ for } \quad k = 0, \ldots, p\nu - 2, \\ \partial_t^{p\nu-1} u_{|t=0} & = u_0 \in \mathcal{A}(B). \end{cases}$$

He proved that its formal power series solution $\widehat{u}$ is $p/(q-p)$-summable in the direction d (also in d' with $d' = d$ mod $(2pi/p)$) iff u_0 extends holomorphically to a domain D containing union of some sectors and has in D at most exponential growth of order at most $q/(p-q)$ loc. uni. in Ω.
He also gave an explicite integral representation of the Borel sum of $\widehat{u}$ in terms of the Meijer function $G_{p,q}^{m,n}$.

Ichinobe also studied the Cauchy problem to the equation

$$\partial_t u = P(t, \partial_z)u, \quad P(t, \partial_z) = \sum_{i,\alpha} a_{i\alpha} t^i \partial_z^\alpha$$

Assuming that the Newton polygon of P has only one slope he proved that the formal solution is k summable if the initial data are holomorphic in a sum of sectors with suitable exponential growth.

From the above papers it follows that formal solutions of non-Kowalevskian PDEs are summable only if the initial data satisfy quite restrictive conditions.

3. Multidimensional case

The study of the multidimensional equations was initiated by Ōuchi [O-02]. He studied the summability of formal solutions to linear PDEs which can be considered as a perturbation of ODEs.

$$(E)\quad \begin{cases} \partial_t^m u \quad + \sum\limits_{(j,\alpha)\in\Lambda} a_{j,\alpha}(t)\partial_t^j \partial_z^\alpha u = f(t,z), \\ \partial_t^i u_{|t=0} = \varphi_i \ \text{ for } \quad i = 0,\ldots,m-1. \end{cases}$$

If $\operatorname{ord}_t a_{j,\alpha} \geq \max(0, j-m+1)$, then the problem has a unique formal solution, which is convergent if $j+|\alpha| \leq m$ for all $(j,\alpha)\in\Lambda$.

Ōuchi defined the Newton polygon $N(E)$ and proved that if
$(j + |\alpha|, \operatorname{ord}_t a_{j,\alpha} - j) \in \operatorname{int} N(E)$ for $(j, \alpha) \in \Lambda$ with $\alpha \neq 0$,
then the formal solution is multisummable in a suitable multidirection; the levels of summability are the slopes of $N(E)$.

In [Y-12] Yamazawa studied the equation

$$\partial_t u = \partial_z^2 u + t(t\partial)^3 u.$$

He proved that if initial data is an entire function of exponential order 2, then the solution is Borel summable in directions $d \notin \{0, \pi\}$. Later he showed that the same conclusion holds for functions of finite exponential order. Motivated by this and similar examples Tahara posed the problem. *Assuming that initial data and f are entire functions of exponential order γ determine minimal γ guaranteeing summability of a solution to (E).*

To solve this problem he and Yamazawa introduced in [TY-13]:
t-Newton polygon $N_t(E)$,
the set of admissible exponents $\mathcal{C}$
and the set of singular directions $\mathcal{Z}$.
They proved that under conditions $(A1)-(A4)$ if initial data and f are entire functions of exponential order $\gamma \in \mathcal{C}$, then the formal solution of (E) is $(k_{p^*}, \ldots, k_1)$-multisummable in any direction $d \notin \mathcal{Z}$; k_i are the slopes of $N_t(E)$.
This result is in accordance with previous ones.

4. Multidimensional heat equation

In the case of the multidimensional heat equation

$$(3)\quad \begin{cases} \partial_t u - \Delta_z u &= 0, \\ u_{|t=0} &= u_0 \in \mathcal{A}(\Omega), \quad \Omega \subset \mathbb{R}^n, \end{cases}$$

where Δ is n-dimensional Laplace operator, conditions for k-summability of formal solutions were obtained by Balser and Malek [BM-04].

However the conditions are stated in terms of some auxiliary function expressed in terms of a formal solutions itself and not in terms of the initial data.

Using the modified Borel transformation Michalik obtained in [M-06] conditions for Borel summability in terms of the initial data. He proved that the formal solution of (3) is Borel summable in a direction d iff the auxiliary function

$$\Phi_n(z,\tau) = \begin{cases} \int_{\partial B(1)} u_0(z+\tau y)dS(y) & \text{if } n \text{ is odd,} \\ \int_{B(1)} \frac{u_0(z+\tau y)}{\sqrt{1-x^2}}dy & \text{if } n \text{ is even} \end{cases}$$

is holomorphic at the origin in z variable and can be analytically continued with respect to τ in sectors in directions $d/2$ and $\pi + d/2$, and this continuation is of exponential order at most 2.

5. Mean values

5.1. Spherical and solid means.

Let Ω be a domain in $\mathbb{R}^n$ and $\mathring{x} \in \Omega$. For $0 < R < \operatorname{dist}(\mathring{x}, \partial\Omega)$ define spherical and solid means of a continuous function $u \in C^0(\Omega)$ by

$$M(u, \mathring{x}; R) = \frac{1}{\sigma(n)R^n} \int_{B(\mathring{x},R)} u(x)dx,$$

$$N(u, \mathring{x}; R) = \frac{1}{n\sigma(n)R^{n-1}} \int_{S(\mathring{x},R)} u(x)dS(x).$$

The relations between $M(u;R)$ and $N(u;R)$ are given by

Lemma 1 *Let $u \in C^0(\Omega)$. Then for any $\mathring{x} \in \Omega$ and $0 < R < \mathrm{dist}(\mathring{x}, \partial\Omega)$,*

$$\Big(\frac{R}{n}\frac{\partial}{\partial R} + 1\Big) M(u, \mathring{x}; R) = N(u, \mathring{x}; R).$$

If $u \in C^2(\Omega)$, then

$$\frac{n}{R}\frac{\partial}{\partial R} N(u, \mathring{x}; R) = M(\Delta u, \mathring{x}; R).$$

5.2. Characterization of real analyticity

Real analyticity of a function can be characterized in terms of its integral means by the Pizzetti series.

Theorem 1 (Mean-value property). ([Ł-12].) *Let $u \in \mathcal{A}(\Omega)$, $\mathring{x} \in \Omega$. Then $M(u, \mathring{x}; R)$ and $N(u, \mathring{x}; R)$ are analytic functions at the origin and for small R,*

$$M(u, \mathring{x}; R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(\mathring{x})}{4^k \left(\frac{n}{2} + 1\right)_k k!} R^{2k}, \tag{4}$$

$$N(u, \mathring{x}; R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(\mathring{x})}{4^k \left(\frac{n}{2}\right)_k k!} R^{2k}. \tag{5}$$

Here $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochhamer symbol.

Theorem 2 (Converse to the mean value property). ([Ł-12].) *Let* $\rho \in C^0(\Omega, \mathbb{R}^+)$, $u \in C^\infty(\Omega)$. *If*

$$\widetilde{N}(x;R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k \left(\frac{n}{2}\right)_k k!} R^{2k}$$

is convergent locally uniformly in $\{(x,R) : x \in \Omega, |R| < \rho(x)\}$, *then* $u \in \mathcal{A}(\Omega)$ *and* $N(u,x;R) = \widetilde{N}(x;R)$ *for* $x \in \Omega$, $R < \min\big(\rho(x), \operatorname{dist}(x, \partial\Omega)\big)$.

5.3. Functions of Laplacian growth.

In order to control the growth of iterated Laplacians of smooth functions Aronszajn, Creese, Lipkin introduced the notion of the Laplacian growth.

Definition [ACL-83]. Let $\varrho > 0$ and $\tau \geq 0$. A function u smooth on $\Omega \subset \mathbb{R}^n$ is of *Laplacian growth* (ϱ, τ) if for every $K \Subset \Omega$ and $\varepsilon > 0$ one can find $C = C(K, \varepsilon) < \infty$ such that for $k \in \mathbb{N}_0$,

$$\sup_{x \in K} |\Delta^k u(x)| \leq C(\tau + \varepsilon)^{2k} (2k)!^{1-1/\varrho}. \tag{6}$$

Definition. ([Boas]) Let $\varrho > 0$ and $\tau \geq 0$. An entire function F is said to be of *exponential growth* (ϱ, τ) if for every $\varepsilon > 0$ one can find C_ε such that for any $R < \infty$

$$\sup_{|z| \leq R} |F(z)| \leq C_\varepsilon \exp\{(\tau + \varepsilon) R^\varrho\}.$$

The exponential growth of an entire function can be also expressed in terms of estimations of its Taylor coefficients.

It appears that a function u of Laplacian growth (ϱ, τ) on Ω is in fact real-analytic on Ω (see [ACL-83, Theorem 2.2 in Chapter II]). So the spherical and solid means $N(u, x; R)$ and $M(u, x; R)$ are expressed by the Pizzetti series valid for $x \in \Omega$ and R small enough. However due to estimation (6) both functions $N(u, x; R)$ and $M(u, x; R)$ can be extended to entire functions of exponential growth.

Theorem 3 ([Ł-12].) *Let* $u \in \mathcal{A}(\Omega)$, $\varrho > 0$ *and* $\tau \geq 0$. *If* u *is of Laplacian growth* (ϱ, τ), *then* $N(u, x; R)$ *and* $M(u, x; R)$ *extend holomorphically to entire functions of exponential growth* $(\varrho, \tau^{\varrho}/\varrho)$ *loc. uni. in* Ω.

Theorem 4 ([Ł-12].) *Let* $u \in \mathcal{A}(\Omega)$. *If* $M(u, x; R)$ *defined for* $x \in \Omega$ *and* $0 \leq R < \operatorname{dist}(x, \partial\Omega)$ *extends to an entire function* $\widetilde{M}(u, x; z)$ *of exponential growth* (ϱ, τ) *loc. uni. in* Ω, *then* u *is of Laplacian growth* $\big(\varrho, (\varrho\tau)^{1/\varrho}\big)$. *Analogous result holds for* $N(u, x; R)$.

5.5. Application to the heat equation

Using the above theorems with $\rho = 2$ we get

Theorem 5 ([Ł-12]). *Let $0 < T \leq \infty$, $u_0 \in \mathcal{A}(\Omega)$. The formal power series solution*

$$\widehat{u}(t,z) = \sum\nolimits_{j=0}^{\infty} \frac{\Delta^j u_0(z)}{j!} t^j \tag{7}$$

of the n-dimensional heat equation (3) is convergent for $|t| < T$ loc. uni. in Ω
iff $N(u_0, z; R)$ and/or $M(u_0, z; R)$ extend to an entire function of exponential growth $(2, 1/(4T))$ loc. uni. in Ω.

Using the above ideas and results from his previous paper [M-06] S. Michalik obtained a characterization of Borel summable solutions of the heat equation (3).

Theorem.([M-12]). *Let $d \in \mathbb{R}$, $U \subset \mathbb{C}^n$ and $\widehat{u}$ be the formal power series solution (7) of the heat equation (3) with $u_0 \in \mathcal{O}(U)$.*
Then the following conditions are equivalent

- $\widehat{u}$ *is* 1*-summable in the direction* d*;*
- $M(u_0; z, R) \in \mathcal{O}^2\big(U \times (\widehat{S}_{d/2} \cup \widehat{S}_{d/2+\pi})\big)$*;*
- $N(u_0; z, R) \in \mathcal{O}^2\big(U \times (\widehat{S}_{d/2} \cup \widehat{S}_{d/2+\pi})\big)$*.*

Furthermore, the 1*-sum of* $\widehat{u}$ *is given by*

$$u^d(t, z) = \frac{1}{(4\pi t)^{n/2}} \int\limits_{(e^{id/2}\mathbb{R})^n} \exp\Big\{\frac{-e^{i\theta}|x|^2}{4t}\Big\} u_0(x + z)dx$$

if the integral is well defined.

5. Heat equation with variable coefficients

The general one dimensional heat equation $\partial_t u - a(z)\partial_z^2 u = \widehat{q}(t,z)$ with a variable coefficient $a(z)$ and inhomogeneity $\widehat{q}(t,z)$ was studied by Balser and Loday-Richaud [BL-09].

Costin, Park and Takei in [CPT-12] studied Borel summability of the IVP

$$\begin{cases} \partial_t u & = a(z)\partial_z^2 u, \\ u(0,z) & = \frac{1}{1+z^2}, \end{cases}$$

where $a(z)$ is a quartic polynomial.

7. Heat equations on manifolds

Let $\mathcal{M}$ be a real analytic manifold of dimension n and $X_1, \ldots, X_d$ real analytic linearly independent vector fields on $\mathcal{M}$. Define a Laplace type operator on $\mathcal{M}$ by $\widetilde{\Delta} = X_1^2 + \cdots + X_n^2$ and consider the IVP

$$(8) \qquad \begin{cases} \partial_t v - \widetilde{\Delta} v = 0, \\ v_{|t=0} = v_0, \qquad v_0 \in \mathcal{A}(\mathcal{M}). \end{cases}$$

The formal power series solution of (8) is given by

$$(9) \qquad \widehat{v}(t,y) = \sum_{k=0}^{\infty} \frac{\widetilde{\Delta}^k v_0(y)}{k!} t^k.$$

It is well known that if vector fields X_i commute,

$$(C) \qquad [X_i, X_j] = 0 \quad \text{for} \quad i, j = 1, \ldots, n,$$

then for a fixed $\mathring{y} \in \mathcal{M}$ one can find a real analytic diffeomorphism $\Phi : \mathbb{R}^n \supset \Omega \xrightarrow{onto} V \subset \mathcal{M}$ s. t. $\mathring{y} \in V = \Phi(\Omega)$ and $\Phi_*^{-1}(X_i) = \frac{\partial}{\partial z_i}$ for $i = 1, \ldots, n$.
Set $B_\Phi(y, R) = \Phi\big(B(z, R)\big)$, $S_\Phi(y, R) = \Phi\big(S(z, R)\big)$, with $z = \Phi^{-1}(y)$, $0 < R < \operatorname{dist}(z, \partial\Omega)$.
Define a measure $\mu_\Phi(A) = \int_{\Phi^{-1}(A)} d\xi$ for $A \subset V$.

Theorem 6 ([Ł-14]). *Let* $0 < T \leq \infty$. *The formal power series solution (9) of the heat type equation (8) is convergent for* $|t| < T$ *loc. uni. in* V *if and only if the solid integral mean*

$$M_\Phi(v_0, y; R) = \frac{1}{\mu_\Phi\big(B_\Phi(y, R)\big)} \int\limits_{B_\Phi(y,R)} v(\eta)\, d\mu_\Phi(\eta)$$

(and/or the spherical integral mean $N_\Phi(v_0, y; R)$*) extends to an entire function of exponential growth* $(2, 1/(4T))$ *loc. uni. in* V.

Theorem 7 ([Ł-14]). *Let $\mathcal{M}$ be a real analytic manifold, $v_0 \in \mathcal{A}(\mathcal{M})$ and $X_1, \ldots, X_n$ real analytic linearly independent commuting vector fields on $\mathcal{M}$. Fix $\mathring{y} \in \mathcal{M}$ and let Φ, Ω, V, B_Φ, μ_Φ and dS_Φ be as in Theorem 6. Set $u_0 = v_0 \circ \Phi$ and assume that u_0 and Φ extend to a complex neighborhood $U \subset \mathbb{C}^n$ of Ω. Then v_0 extends to the neighborhood $\Phi(U)$ of V in the complexification of $\mathcal{M}$.*

Let $d \in \mathbb{R}$ and let $\widehat{v}$ be the formal solution (9) *of the heat type equation* (8)*. Then TFCAE:*

1. *$\widehat{v}$ is Borel summable in d loc. uni. in $\Phi(U)$;*
2. *$M_\Phi(v_0; z, R)$ extends to $\Phi(U) \times \big(D_\epsilon \cup S(d/2, \epsilon) \cup S(d/2 + \pi, \epsilon)\big)$ with $0 < \epsilon$ and for any $U_1 \Subset U$, $0 < \epsilon_1 < \epsilon$ and $R \in S(d/2, \epsilon_1) \cup S(d/2 + \pi, \epsilon_1)$, $\sup_{z \in \Phi(U_1)} |M_\Phi(v_0; z, R)| \leq Ae^{B|R|^2}$;*
3. *The same holds for $N_\Phi(v_0; z, R)$.*

8. Nonlinear equations

The study of summability of formal solutions to nonlinear partial differential equations is just starting. Ōuchi in [O-06] studied formal solutions for a class of singular partial differential equations with polynomial nonlinearity and proved their summability under the assumption of vanishing initial data.

8.1. Burgers equation

In [Ł-09] we considered the IVP for the Burgers equation

$$(BE) \qquad \begin{cases} \partial_t u - \partial_z^2 u &= \partial_z (u^2), \\ u_{|t=0} &= u_0. \end{cases}$$

The formal power series solution is given by

$$\widehat{u}(t,z) = \sum_{k=0}^{\infty} \frac{u_k(z)}{k!} t^k,$$

where

$$
\begin{aligned}
u_{k+1} &= \partial^2 u_k + v_k,\\
v_k &= \sum_{\kappa\in\mathbb{N}_0^2,\, \kappa_1+\kappa_2=k} \partial\big(u_{\kappa_1}u_{\kappa_2}\big).
\end{aligned}
$$

Applying the Cole-Hopf transformation

$$
u(t,z) \mapsto v(t,z) = \exp\Big\{\int^z u(t,y)dy\Big\}
$$

which transforms (BE) into the heat equation and its inverse $v(t,z) \mapsto u(t,z) = \big(\ln v(t,z)\big)'_z$ we proved

Theorem 8 [Ł-09]. *Let $u_0 \in \mathcal{A}(B)$, B - a ball. If the formal power series solution of (BE) is convergent loc. uni. in B, then u_0 extends to a meromorphic function of the form*

$$(10) \quad u_0(z) = 2az + b + \sum_{n=1}^{\infty} \left(\frac{1}{z - z_n} + \frac{1}{z_n} + \frac{z}{z_n^2} \right),$$

where $a, b \in \mathbb{C}$ and

$\{z_n\}_{n\in\mathbb{N}}$ is a sequence of $z_n \in \mathbb{C}^* \cup \{\infty\}$ with nondecreasing modulus such that

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{2+\varepsilon}} < \infty \quad \textit{for} \quad \textit{any} \quad \varepsilon > 0. \tag{11}$$

Conversely, if u_0 extends to a meromorphic function of the form (10) and (11) holds, then the formal solution of (BE) is convergent in a nbh. of $\{0\}$.

Theorem 9 [Ł-09]. *Let $u_0 \in \mathcal{A}(B)$ and $d \in \mathbb{R}$. If the formal power series solution of (BE) is Borel summable in the direction d loc. uni. in a nbh. of $\{0\}$, then u_0 extends analytically to a function meromorphic on a domain*

$$D(d,\varepsilon) \supset S(d/2,\varepsilon) \cup S(d/2+\pi,\varepsilon)$$

with some $\varepsilon > 0$ which has in $D(d,\varepsilon)$ at most simple poles with residua in $\mathbb{N}$.

Conversely, if u_0 extends to a meromorphic function on $D(d,\varepsilon)$ of the form

$$u_0(z) = \sum_{n=1}^{\infty} \Big(\frac{1}{z - z_n} + \frac{1}{z_n} + \frac{z}{z_n^2} \Big) + v(z),$$

where $0 \neq z_n \in D(d,\varepsilon)$ satisfy (11),
v is holomorphic on $D(d,\varepsilon)$, $|v(z)| \leq a|z| + b$, then the formal power series solution of (BE) is Borel summable in the direction d loc. uni. in a nbh. of (0).

9. Characterization of real analyticity

Theorem 10 ([Ł-18]). *Let $u \in C^0(\Omega)$.
If there exist functions $u_k \in C^0(\Omega)$ for $k \in \mathbb{N}_0$ and $\epsilon \in C^0(\Omega, \mathbb{R}_+)$ such that*

$$M(u; x, R) = \sum_{k=0}^{\infty} u_k(x) R^k,$$

locally uniformly in $\{(x, R) : x \in \Omega, |R| < \epsilon(x)\}$, then u is real analytic on Ω and for $l \in \mathbb{N}_0$, $u_{2l+1} = 0$ and $u_{2l} = \left(4^l \left(\frac{n}{2} + 1\right)_l l!\right)^{-1} \cdot \Delta^l u$

We proposed a definition of analytic functions on MMS.

Definition ([Ł-18]). Let (X, ρ, μ) be a metric measure space with a metric ρ and a Borel regular measure μ which is positive on open sets and finite on bounded sets. Let Ω be an open subset of X. For any $x \in \Omega$ and $0 < R < \operatorname{dist}(x, \partial\Omega)$ define a solid mean of a continuous function $u \in C^0(\Omega)$ by

$$M_X(u;\, x, R) = \frac{1}{\mu(B_\rho(x, R))} \int_{B_\rho(x,R)} u(y)\, d\mu(y).$$

Definition ([Ł-18]). Let (X, ρ, μ) be a metric measure space and Ω be an open subset of X. Let $u \in C^0(\Omega, \mathbb{C})$.
We say that u is (X, ρ, μ)-*analytic on* Ω and write $u \in \mathcal{A}_X(\Omega, \rho, \mu)$ if there exist functions $u_l \in C^0(\Omega)$ for $l \in \mathbb{N}_0$ and $\epsilon \in C^0(\Omega, \mathbb{R}_+)$ such that

$$M_X(u;\, x, R) = \sum_{l=0}^{\infty} u_l(x) R^l.$$

locally uniformly in $\{(x, R) : x \in \Omega, |R| < \epsilon(x)\}$.

Theorem 11 ([Ł-20]). *Let (X, ρ) be a proper, locally uniquely geodesic, metric space satisfying property (P) and let μ be a Borel measure that is finite on compact sets and positive on open sets. Let Ω be a connected open subset of X and let $u \in C^0(\Omega)$ be an (X, ρ, μ)-analytic function on Ω.*
If u vanishes on a nonempty open set $U \subset \Omega$, then $u \equiv 0$ on Ω.

References

[ACL-83] N. Aronszajn, T. M. Creese and L. J. Lipkin, *Polyharmonic Functions*, Clarendon Press, Oxford, 1983.

[B-99] W. Balser, *Divergent solutions of the heat equation: on an article of Lutz, Miyake and Schäfke*, Pacific J. Math. 188 (1999), 53–63.

[BM-99] W. Basler, M. Miyake, *Summability of formal solutions of certain partial differential equations*, Acta Sci. Math. (Szeged), 65 (1999), 543–551.

[B-02] W. Balser, *Summability of formal power series solutions of partial differential equations with constant coefficients*, in: *Proc. Intern. Conf. on Differential and Functional Differential Equations*, Moscow (2002).

[B-04] W. Balser, *Multisummability of formal power series solutions of partial differential equations with constant coefficients*, J. Differential Equations 201 (2004), 63–74.

[B-05] W. Balser, *Asymptotic existence theorems for formal power series whose coefficients satisfy certain partial differential recursions*, J. Differential Equations 209 (2005), 442–457.

[BL-09] W. Balser and M. Loday-Richaud, *Summability of solutions of the heat equation with inhomogeneus thermal conductivity in two variables*, Adv. Dyn. Sys. Appl. 4 (2009), 159–177.

[BM-04] W. Balser and S. Malek, *Formal solutions of the complex heat equation in higher spatial variables*, in: *Proc. Global and Asymptotic Analysis of Differential Equations in the Complex Domain*, Kôkyûroku RIMS 1367 (2004), 87–94.

[Boas] R. P. Boas Jr, *Entire Functions*, Academic Press, New York, 1954.

[CPT-12] O. Costin, H. Park, Y. Takei, *Borel summability of the heat equation with variable coefficients*, J. Differential Equations 252 (2012), 3076–3092.

[H-06] M. Hibino, *Borel summability of divergent solutions for singular first-order partial differential equations with variable coefficients*, J. Differential Equations 227 (2006), 499–533, 534–563.

[I-01] K. Ichinobe, *The Borel sum of divergent Barnes hypergeometric series and its application to a partial differential equation*, Publ. Res. Inst. Math. Sci. 37 (2001), 91-117.

[I-03] K. Ichinobe, *Integral representation for Borel sum of divergent solution to a certain non-Kowalevski type equation*, Publ. Res. Inst. Math. Sci. 39 (2003), 657-693.

[Kow-1875] S. Kowalevski, *Zur Theorie der partiellen Differentialgleichungen*, J. Reine Angew. Math. 80 (1875), 1–32.

[LMS-99] D. A. Lutz, M. Miyake and R. Schäfke, *On the Borel summability of divergent solutions of the heat equation*, Nagoya Math. J. 154 (1999), 1–20.

[Ł-09] G. Łysik, *Borel summable solutions of the Burgers equation*, Ann. Polon. Math. 95 (2009), 187–197.

[Ł-12] G. Łysik, *Mean-value properties of real analytic functions*, Arch. Math. (Basel) 98 (2012), 61–70.

[Ł-14] G. Łysik, *The Borel summable solutions of heat operators on a real analytic manifold*, J. Math. Anal. Appl. 410 (2014), 117–123.

[Ł-18] G. Łysik, *A characterization of real analytic functions*, Ann. Acad. Sci. Fenn., Math. **43** (2018), 1–8.

[Ł-20] G. Łysik, *A uniqueness property for analytic functions on metric measure spaces*, Proc. AMS, **148** (2020), 1679–1688.

[M-06] S. Michalik, *Summability of divergent solutions of the n-dimensional heat equation*, J. Differential Equations 229 (2006), 353–366.

[M-10] S. Michalik, *On the multisummability of divergent solutions of linear partial differential equations with constant coefficients*, J. Differential Equations 249 (2010), 551–570.

[M-12] S. Michalik, *On Borel summable solutions of the multidimensional heat equation*, Ann. Polon. Math. 105 (2012), 167–177.

[Miy-99] M. Miyake, *Borel summability of divergent solutions of the Cauchy problem to non Kovaleskian equations*, in: *Proc. of the Conf. on Partial Dif-*

ferential Equations and Their Applications, (edited by H. Chen and L. Rodino), World Scientific, 1999, pp. 225–239.

[O-02] S. Ōuchi, *Multisummability of formal solutions of some linear partial differential equations*, J. Differential Equations 185 (2002), 513–549.

[O-06] S. Ōuchi, *Multisummability of formal power series solutions of nonlinear partial differential equations in complex domains*, Asymptot. Anal. 47 (2006), 187–225.

[TY-13] H. Tahara and H. Yamazawa, *Multisummability of formal solutions to the Cauchy problem for some linear partial differential equations*, J. Differential Equations 255 (2013), 3592–3637.

[Y-12] H. Yamazawa, *Borel summability for a formal solutions of* $\frac{\partial}{\partial t}u(t,x) = \left(\frac{\partial}{\partial x}\right)^2 u(t,x) + t\left(t\frac{\partial}{\partial t}\right)^3 u(t,x)$, in: *Formal and Analytic Solutions of Differential and Difference Equations*, Banach Center Publ. vol. 97 (2012), 161–168.

Thank you for your attention